Lec 8: 09/21/2010

Equations of Stellar Evolution:

The basic differential equations for a spherically symmetric star are:

$$\frac{\partial P}{\partial M_r} = -\frac{G M_r}{4\pi r^4} - \frac{1}{4\pi r^2}\frac{\partial^2 r}{\partial t^2} \qquad *$$

$$\frac{\partial L_r}{\partial M_r} = \varepsilon - \varepsilon_\nu - c_p \frac{\partial T}{\partial t} + \frac{\delta}{\rho}\frac{\partial P}{\partial T} \qquad **$$

$$\frac{\partial T}{\partial M_r} = -\frac{G M_r T}{4\pi r^4 P}\nabla \qquad ***$$

$$\frac{\partial X_i}{\partial t} = \frac{m_i}{\rho}\left(\sum_j r_{ji} - \sum_k r_{ik}\right) \quad i = 1, \ldots, I \qquad ****$$

Here M_r is the mass contained within radius r and is related to mass density ρ according to:

$dM_r = 4\pi r^2 \rho \, dr$

X_i is the mass fraction of element i ($\sum_i X_i = 1$). c_p is the specific heat capacity at constant pressure defined as:

$$c_p \equiv \left(\frac{dq}{dT}\right)_p = \left(\frac{\partial u}{\partial T}\right)_p + P\left(\frac{\partial v}{\partial T}\right)_p$$

dq is the heat added per unit mass, u is the internal energy per unit mass, and v is the specific volume per unit mass. Note that $du = dq - P\,dv$ from the first law of thermodynamics.

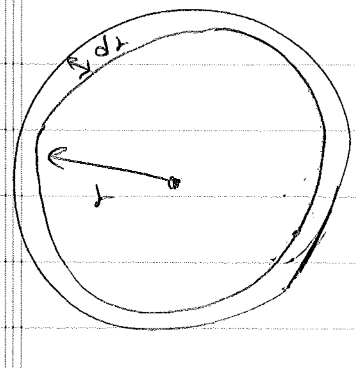
δ is defined as follows:

$$\delta \equiv -\left(\frac{\partial \ln \rho}{\partial \ln T}\right)_P = \frac{T}{v}\left(\frac{\partial v}{\partial T}\right)_P$$

ε and ε_ν are nuclear energy released and escaped energy in neutrinos per unit mass.

Now lets have a closer look at these equations.

Equation a can be easily derived by considering a mass shell at a radial distance r from the center.



The net force on the shell is the sum of the force from pressure gradient $\frac{-\partial P}{\partial r} 4\pi r^2$ and gravitational force $\frac{G M_r}{r^2} 4\pi r^2 dr \rho$. According to Newton's 2nd law the net force is equal to $4\pi r^2 dr \rho \frac{\partial^2 r}{\partial t^2}$.

To understand equation, we note that the difference between the luminosity at $r+dr$ and at r is due to energy production from nuclear burning (ε), energy escaped in the form of neutrinos ($-\varepsilon_\nu$) and the heat deposited within the shell (dq).

From the first law of thermodynamics:

$$dq = du + P dV = \left(\frac{\partial u}{\partial T}\right)_P dT + \left(\frac{\partial u}{\partial P}\right)_T dP + P\left(\frac{\partial V}{\partial T}\right)_P dT + P\left(\frac{\partial V}{\partial P}\right)_T dP$$

$$= \underbrace{\left[\left(\frac{\partial u}{\partial T}\right)_P + P\left(\frac{\partial V}{\partial T}\right)_P\right]}_{C_P} dT + \left[\left(\frac{\partial u}{\partial P}\right)_T + P\left(\frac{\partial V}{\partial P}\right)_T\right] dP$$

We recall the definition of entropy $ds = \frac{dq}{T}$, which leads to:

$$ds = \frac{1}{T}du + \frac{P}{T}dv = \frac{1}{T}\left[\left(\frac{\partial U}{\partial T}\right)_v dT + \left(\frac{\partial U}{\partial v}\right)_T dv\right] + \frac{P}{T}dv =$$

$$\frac{1}{T}\left(\frac{\partial U}{\partial T}\right)_v dT + \left[\frac{1}{T}\left(\frac{\partial U}{\partial v}\right)_T + \frac{P}{T}\right]dv$$

Using the fact that $\frac{\partial^2 s}{\partial v\,\partial T} = \frac{\partial^2 s}{\partial T\,\partial v}$, we find:

$$\frac{\partial}{\partial v}\left[\frac{1}{T}\left(\frac{\partial U}{\partial T}\right)_v\right] = \frac{\partial}{\partial T}\left[\frac{1}{T}\left(\frac{\partial U}{\partial v}\right)_T + \frac{P}{T}\right] \Rightarrow \cancel{\frac{1}{T}\frac{\partial^2 U}{\partial v\,\partial T}}$$

$$= -\frac{1}{T^2}\left(\frac{\partial U}{\partial v}\right)_T + \cancel{\frac{1}{T}\frac{\partial^2 U}{\partial T\,\partial v}} - \frac{1}{T^2}P + \frac{1}{T}\left(\frac{\partial P}{\partial T}\right)_v \Rightarrow T\left(\frac{\partial P}{\partial T}\right)_v =$$

$$P + \left(\frac{\partial U}{\partial v}\right)_T \Rightarrow P = T\left(\frac{\partial P}{\partial T}\right)_v - \left(\frac{\partial U}{\partial v}\right)_T$$

Replacing this last expression for P in the last line on the previous page, we get:

$$dq = c_p\,dT + \left[\cancel{\left(\frac{\partial U}{\partial P}\right)_T} + T\left(\frac{\partial P}{\partial T}\right)_v\left(\frac{\partial v}{\partial P}\right)_T - \cancel{\left(\frac{\partial U}{\partial v}\right)_T\left(\frac{\partial v}{\partial P}\right)_T}\right]dP$$

We now use the following identity:

$$\left(\frac{\partial P}{\partial T}\right)_v\left(\frac{\partial v}{\partial P}\right)_T\left(\frac{\partial T}{\partial v}\right)_P = -1$$

To derive:

$$dq = c_p\,dT - T\left(\frac{\partial v}{\partial T}\right)_P dP$$

From definition of δ, and noticing that $\rho v = 1$, we then arrive at equation ** on page 67.

Equation *** describes temperature gradient, and how it is related to the flux of energy (encoded in ∇). Two cases are possible: energy transfer by radiation, and energy transfer by convection.

Assuming radiative energy transfer is dominant, the energy flux at radial distance r $F_r = \frac{L_r}{4\pi r^2}$ results in a momentum flux $\frac{F_r}{c}$. Momentum scattered per unit volume per second is $\frac{n\sigma}{c} F_r$, where n is the number density of scatterers and σ is the scattering cross-section.

One defines opacity κ according to $n\sigma = \kappa\rho$.

We then have:

$$\frac{\kappa\rho}{c}\frac{L_r}{4\pi r^2} = -\frac{\partial P_{rad}}{\partial r} = -\frac{\partial}{\partial r}\left(\frac{1}{3}\rho_{rad}\right) = -\frac{1}{3}\frac{\partial}{\partial r}\left(aT^4\right)$$

($a = 7.57 \times 10^{-15} \frac{erg}{cm^3 K^4}$: radiation-density constant)

$$\Rightarrow \frac{\partial T}{\partial r} = \frac{-3}{4} \frac{\kappa \rho}{c} \frac{L_r}{4\pi r^2} \frac{1}{aT^3} \Rightarrow \frac{\partial T}{\partial M_r} = \frac{-3\kappa \rho L_r}{64\pi^2 c r^4 a T^3}$$

$$\boxed{\Rightarrow \frac{\partial T}{\partial M_r} = \frac{-GM_r T}{4\pi r^4 P} \nabla_{rad} \, , \quad \nabla_{rad} \equiv \frac{3}{16\pi a c G} \frac{\kappa L_r P}{M_r T^4}}$$

This is equation ** with $\nabla = \nabla_{rad}$.

On the other hand, if convective energy transfer dominates over radiative transfer, we have:

$$\nabla = \nabla_{ad} \, , \quad \nabla_{ad} \equiv \frac{\partial \ln T}{\partial \ln P}$$

For an ideal gas, it can be shown that $\nabla_{ad} = \frac{\gamma - 1}{\gamma}$, where γ is adiabatic exponent. Note that for adiabatic condition:

$P v^{\gamma} = \text{Const}$, $\quad P v \propto T \longrightarrow$ equation of state

$$\Rightarrow dP \, v^{\gamma} + \gamma v^{\gamma - 1} P \, dv = 0 \, , \quad dT \propto P \, dv + v \, dP$$

Thus:

$$\frac{dT}{T} = \frac{P\,dv + v\,dP}{Pv} = \frac{dv}{v} + \frac{dP}{P} = \frac{dP}{P} - \frac{1}{\gamma}\frac{dP}{P} \Rightarrow$$

$$\frac{dT}{T} = \frac{\gamma - 1}{\gamma} \frac{dP}{P} \Rightarrow \boxed{\nabla_{ad} = \frac{\gamma - 1}{\gamma}}$$

Radiation will be dominant if $\nabla_{rad} \leq \nabla_{ad}$, while for $\nabla_{ad} \leq \nabla_{rad}$ convection will take over.

Finally, equation ~~****~~ on page 67 gives time evolution of the mass fraction of element i. $\sum_j r_{ji}$ accounts for all nuclear reactions that yield element i, while $\sum_k r_{ik}$ takes into account of all reactions that destroy element i.

Obviously, the system of partial differential equations given on page 67 are very complicated. To make things simpler, we make the following assumptions:

1) The star is assumed to be time independent. That is, the time scale over which any of the macroscopic physical properties of the star changes is significantly longer than the time scales involved in estabishing and maintaining mechanical and thermal equilibrium inside

the star. Conventionally such a star is called a zero-age main-sequence (ZAMS) star.

2) As a corollary, we ignore any gradients in the composition of the interior and take it to be chemically homogeneous.

3) The star is spherically symmetric and all effects of rotation and magnetic field on the structure is ignored. Equivalently, the kinetic energy due to rotation and magnetic energy is assumed to be negligible compared with the thermal and gravitational energy.

4) Due to the lack of a fundamental theory of convective transport, the convective energy transport is treated in a very simplified way, namely the criterion $\nabla_{ad} < \nabla_{rad}$ is used.

5) Simple boundary conditions are used near the surface of the star, and phenomena like stellar winds

are ignored.

6) The gravitational field is described by Newtonian theory and all general relativistic effects are negligible.

The first assumption, the star being time independent, results in a great simplification by making partial differential equations become ordinary differential equations, with r being the only relevant variable. We then have;

$$\boxed{\left\{\begin{aligned} \frac{dP}{dM_r} &= -\frac{GM_r}{4\pi r^4} \\ \frac{dL_r}{dM_r} &= \varepsilon \qquad \text{(ignoring neutrinos)} \\ \frac{dT}{dM_r} &= \begin{cases} \dfrac{-3\kappa}{64\pi^2 ac}\left(\dfrac{1}{T^3}\right)\left(\dfrac{L_r}{r^4}\right) & \nabla_{rad} \leq \nabla_{ad} \\ \dfrac{-\nabla_{ad}}{4\pi}\left(\dfrac{T}{P}\right)\left(\dfrac{GM_r}{r^4}\right) & \nabla_{ad} \leq \nabla_{rad} \end{cases} \end{aligned}\right.}$$

The boundary conditions are;

$$\boxed{L_r = 0 \text{ at } M_r = 0 \text{ (center)}, \quad P = T = 0 \text{ at } M_r = M \text{ (surface)}}$$

Because of the complex nature of these equations, it is not easy to rigorously determine existence and uniqueness of solutions. Intuitively, one expects to find no solutions for very low mass or for arbitrary chemical composition. Explicit (numerical) integration shows that solutions do exist for $M \gtrsim 0.1\, M_\odot$ with $X \simeq 0.7 - 1.0$ (X being the mass fraction of Hydrogen). In principle, it is possible to have situations in which the solution is not unique. However, such situations usually turn out to be physically unacceptable, and hence not relevant. In physically realistic situations, the solution may be taken to be unique.